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Analyticity and Discrete Maximal Regularity on L_p-Spaces

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L_p: If T is powerbounded on L_p and L_q as well as analytic on L_p, then T is power-bounded and analytic on L_r for all r strictly between p and q. This is a discrete analogue of the well-known corresponding result for analytic semigroups (e^{tA}).

As recently shown by the author, the analyticity of T is a necessary condition for the maximal regularity of the discrete time evolution equation $u_{n+1} - Tu_n = f_n$ for all $n \in \mathbb{Z}_+$, $u_0 = 0$. In the second part of this paper we establish the following two sufficient conditions for its maximal regularity: T is a subpositive analytic contraction, or T is an integral operator satisfying certain Poisson bounds. These results are discrete analogues of the corresponding results for the maximal regularity of the evolution equation $u'(t) - Au(t) = f(t)$ for all $t \in \mathbb{R}_+$, $u(0) = 0$, due to Lamberton, Weis, Coulhon and Duong and Hieber and Prüss. For the Poisson bound result of Coulhon and Duong and Hieber and Prüss we give a slight improvement and a short proof. © 2001 Academic Press

1. INTRODUCTION AND MAIN RESULTS

The well-known problem of maximal L_p-regularity for continuous time evolution equations is the following. Let X be a Banach space and A the generator of a bounded analytic semigroup on X. We consider the evolution equation

$$u'(t) - Au(t) = f(t) \qquad \text{for all} \quad t \in \mathbb{R}_+, \quad u(0) = 0. \tag{1}$$

One says that A *has maximal regularity* if for every right hand side $f \in L_p(\mathbb{R}_+; X)$ the solution u satisfies $u' \in L_p(\mathbb{R}_+; X)$. See [W2, Section 1] for a recent survey on maximal regularity.

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In [B1] we considered the maximal regularity problem for the following natural discrete time evolution equation with values in X:

$$u_{n+1} - Tu_n = f_n \qquad \text{for all} \quad n \in \mathbb{Z}_+, \quad u_0 = 0$$

We say that the powerbounded operator T *has discrete maximal regularity* if for every right hand side $f \in l_p(\mathbb{Z}_+; Z)$ the discrete derivative $(u_{n+1} - u_n)$ of the solution u belongs to $l_p(\mathbb{Z}_+; X)$.

This discrete version of the maximal regularity problem was formulated and indicated to the author by T. Coulhon.

It was shown in [B1] that if T has discrete maximal regularity then T is *analytic* in the sense of [C-SC]:

$$\{n(T-I)\, T^n;\, n \in \mathbb{N}\} \text{ is bounded.} \tag{2}$$

This notion is a discrete analogue of the property "$\{tAe^{tA};\, t > 0\}$ is bounded" which characterizes the analyticity of a bounded semigroup $(e^{tA})_{t \geqslant 0}$. Moreover, the following characterization was proved in [B1]:

Theorem A. *Let X be a UMD space and let $T \in \mathfrak{L}(X)$ be powerbounded and analytic. Then the following conditions are equivalent*:

(a) *T has discrete maximal regularity.*

(b) *$\{(\lambda - 1)\, R(\lambda, T);\, \lambda \in 1 + i\mathbb{R},\, \lambda \neq 1\}$ is R-bounded.*

(c) *$\{T^n, (T-I)\, T^n;\, n \in \mathbb{N}\}$ is R-bounded.*

(d) *$A := T - I$ has maximal regularity.*

Here we used the notion of R-boundedness which was already implicitly used in [Bou] and was introduced in [BG]. A set $\tau \subset \mathfrak{L}(X)$ is called *R-bounded* if there is a constant C such that we have for all $n \in \mathbb{N}$, $T_1, ..., T_n \in \tau$ and $f_1, ..., f_n \in X$:

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t)\, T_j f_j \right\| dt \leqslant C \int_0^1 \left\| \sum_{j=1}^n r_j(t)\, f_j \right\| dt,$$

where (r_j) is a sequence of independence symmetric $\{1, -1\}$-valued random variables on $[0, 1]$, e.g. the Rademacher functions.

In this paper we study the analyticity and the discrete maximal regularity for the case of operators T on $X = L_p$.

Our first result is the discrete analogue of the well-known fact that a strongly continuous semigroup (e^{tA}) which is bounded analytic on L_p and bounded on L_q is bounded analytic on L_r for all r strictly between p and q. This gives the answer to a question posed in [C-SC].

THEOREM 1.1. *Let $p,\ q \in [1, \infty]$ and $T \in \mathfrak{L}(L_p)$ be powerbounded and analytic. If T is powerbounded on L_q then T is powerbounded and analytic on L_r for all r strictly between p and q.*

The rest of the paper concerns the verification of discrete maximal regularity for operators $T \in \mathfrak{L}(L_p)$. This will be achieved by developing further the techniques introduced by Weis in [W2] which are based on domination, interpolation and R_q-boundedness.

We give slight improvement (see Theorem 4.3 below), a short proof and—above all—the following discrete analogue of a result due to Coulhon and Duong [CD] (see also [HP]) on the maximal regularity of semigroups (e^{tA}) on L_p having an integral kernel which satisfies certain Poisson bounds. We use some standard notations for the setting of spaces (Ω_1, μ, d) of homogenous type [CD] [DR].

THEOREM 1.2. *Let Ω_1 be a space of homogeneous type and dimension $D > 0$, i.e.*

$$V_{\Omega_1}(x, \lambda r) \leqslant C\lambda^D V_{\Omega_1}(x, r) \qquad \text{for all} \quad x \in \Omega, \quad \lambda \geqslant 1, \quad r > 0.$$

Let Ω be a measurable subset of Ω_1 and let the powers T^n of $T \in \mathfrak{L}(L_2(\Omega))$ have integral kernels $p_n(x, y)$ satisfying the following Poisson bounds:

$$|D^k p_n(x, y)| \leqslant n^{-k} V_{\Omega_1}(x, n^{1/m})^{-1}\, g(d(x, y)^m\, n^{-1}) \qquad \text{for all } n \in \mathbb{N}, \quad (3)$$

for $k = 0,\ 1$, some $m > 0$ and a bounded decreasing function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\sum_k k^{D-1} g(k^m) < \infty$. Then, for all $p \in (1, \infty)$, T is powerbounded and analytic on $L_p(\Omega)$ and has discrete maximal regularity on $L_p(\Omega)$.

Here $D^k p_n$ denotes the kernel of the operator $(T - I)^k\, T^n$. In the case $g(r) = C \exp(-br^{1/(m-1)})$ of Gaussian-type bounds, such estimates are quite common for $m = 2$ [H-SC]; for $m \neq 2$ they appear on the so-called graphical Sierpinski gaskets and related graphs with fractal structure [J] [BB]. For this Gaussian-type case, it was shown in [B2] that, under rather general conditions, line (3) holds for $k = 1$ if it holds for $k = 0$.

Furthermore, we present a criterion for R_∞-boundedness (see Theorem 5.1 below) which is based on the transference principle of R. R. Coifman and G. Weiss [CW] and generalizes L. Weis' approach in [W2] using maximal estimates.

As an application of our criterion, we prove the following discrete analogue of a result due to Weis [W2] and (in a slightly weaker version) Lamberton [L] saying that the operator A on L_p has maximal regularity if (e^{tA}) is a subpositive analytic contractive semigroup.

THEOREM 1.3. *Let* $p \in (1, \infty)$ *and* $T \in \mathfrak{L}(L_p)$ *be a subpositive analytic contraction. Then* T *has discrete maximal regularity.*

Here the subpositivity of a contraction T [resp. of a contractive C_0-semigroup (e^{tA})] on L_p is defined by the existence of a dominating positive contraction S [resp. of a dominating positive contractive C_0-semigroup (e^{tB})], i.e.

$$|Tf| \leqslant S\,|f| \qquad [\text{resp. } \forall t > 0 : |e^{tA} f| \leqslant e^{tB}\,|f|] \qquad \text{for all} \quad f \in L_p.$$

Our Theorem 1.3 shows that e.g. all Markov operators T have discrete maximal regularity on L_p for all $p \in (1, \infty)$; this includes random walks on graphs. Note that by Theorem 1.2 we can also treat the case of non-positive integral kernels.

In [B1] we gave the following proof of Theorem 1.3. Under the given hypotheses, $(e^{t(T-I)})$ is a subpositive analytic contractive semigroup on L_p, hence $A := T - I$ has maximal regularity due to the result of Lamberton and Weis we just mentioned. Thus condition (d) of Theorem A is satisfied and we deduce the discrete maximal regularity of T.

Here we will verify directly the condition (b) of Theorem A by using the dilation theorem of Akcoglu and Sucheston [AS] and not (as in Weis' proof of the continuous time result) its semigroup version due to Fendler [F] which is much more complicated.

The plan of this paper is as follows. In Section 2 we give resolvent conditions concerning the powerboundedness and the analyticity of an operator T on a Banach space X. Afterwards, these results will be applied for the case $X = L_p$ to give the proof of Theorem 1.1.

In Section 3 we give Weis' motivation and definition of R_q-boundedness and adjust his interpolation techniques [W2] to our purposes.

Section 4 establishes our results (e.g. Theorem 1.2) on (discrete) maximal regularity for integral operators dominated by Poisson-type integral operators or—as pointed out in [DR], more generally—by the Hardy–Littlewood maximal operator.

In Section 5 we prove the announced transference principle for R_∞-boundedness. Subsequently, we show that our Theorem 1.3 and the Maximal Ergodic Theorem can be obtained as applications.

2. POWERBOUNDEDNESS AND ANALYTICITY

2.1. *Resolvent Conditions*

Let X be a Banach space and $T \in \mathfrak{L}(X)$. In this section we give resolvent conditions on T connected to the powerboundedness and the analyticity of T.

We start by recalling some well-known resolvent conditions for a *continuous time semigroup* (e^{tA}) on X which we assume to be strongly continuous [P]. Firstly, if (e^{tA}) is a bounded semigroup, then

$$\{\lambda R(\lambda, A);\ \lambda \in \Sigma_{\frac{\pi}{2}-\delta}\} \text{ is bounded for all } \delta > 0. \tag{4}$$

Here and in the sequel, Σ_γ denotes the open sector $\{z;\ |\arg(z)| < \gamma\}$. Secondly, (e^{tA}) is a bounded analytic semigroup if and only if

$$\{\lambda R(\lambda, A);\ \lambda \in \Sigma_{\frac{\pi}{2}+\delta}\} \text{ is bounded for some } \delta > 0. \tag{5}$$

The following two propositions show that resolvent conditions on a *discrete time semigroup* (T^n) to be bounded (and analytic) are suggestive modifications of (4) and (5) involving the Cayley transform $\mathscr{C}(z) := \frac{z-i}{z+i}$.

Proposition 2.1. *Let X be a Banach space and $T \in \mathfrak{L}(X)$ be powerbounded. Then*

$$\{(\lambda - 1)\, R(\lambda, T);\ \lambda \in \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}-\delta} + \varepsilon])\} \textit{ is bounded for all } \delta, \varepsilon > 0.$$

Proposition 2.2. *Let X be a Banach space and $T \in \mathfrak{L}(X)$. Then the following are equivalent*:

(a) *T is powerbounded and analytic.*

(b) *$(e^{t(T-I)})$ is a bounded analytic semigroup and $\sigma(T) \subset \mathbb{D} \cup \{1\}$.*

(c) *$\{(\lambda - 1)\, R(\lambda, T);\ \lambda \in \overline{\mathbb{D}}^c \cup (1 + \Sigma_{\frac{\pi}{2}+\delta})\}$ is bounded for some $\delta > 0$.*

(d) *$\{(\lambda - 1)\, R(\lambda, T);\ \lambda \in \overline{\mathbb{D}}^c\}$ is bounded.*

(e) *$\{(\lambda - 1)\, R(\lambda, T);\ \lambda \in \mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta})\}$ is bounded for some $\delta > 0$.*

(f) *$\{(\lambda - 1)\, R(\lambda, T);\ \lambda \in \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}+\delta} - \varepsilon])\}$ is bounded for some δ, $\varepsilon > 0$.*

(g) *T is powerbounded and $\{(\lambda - 1)\, R(\lambda, T);\ |\lambda| = 1, \lambda \neq 1\}$ is bounded.*

Here $\mathbb{D}$ is the open unit disk in $\mathbb{C}$. The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) are essentially due to Nevanlinna [N1]; see [B1, Theorem 2.3] for a short proof. The equivalence of (d) was remarked in [NZ] and [Ly]. I owe to T. Coulhon the idea of using the Cayley transform to connect discrete and continuous time conditions.

The following lemma prepares the proofs of the preceding results and describes the images of sectors $-i\Sigma_\delta$ under the Cayley transform. For the rest of this section, we denote by $B(z, r)$ closed balls in $\mathbb{C}$.

LEMMA 2.3. *Let $\delta \in [0, \frac{\pi}{2})$ and $\varepsilon > 0$.*

(a) $\mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta}) = B(i\tan(\delta), \cos(\delta)^{-1})^C \cup B(-i\tan(\delta), \cos(\delta)^{-1})^C$

(b) $\mathscr{C}(-i\Sigma_{\frac{\pi}{2}-\delta}) = B(i\tan(\delta), \cos(\delta)^{-1})^C \cap B(-i\tan(\delta), \cos(\delta)^{-1})^C$

(c) *There exists $r > 0$ such that*

$$\mathscr{C}(-i\Sigma_{\frac{\pi}{2}-\delta})\backslash B(-1, r) \supset \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}-\delta} + \varepsilon]).$$

(d) *If $\delta > 0$ then there exist $\delta', \varepsilon' > 0$ such that*

$$\mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta}) \cup B(-1, \varepsilon) \supset \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}+\delta'} - \varepsilon']).$$

Proof. (a) and (b) are direct consequences of the fact that, for each $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the Cayley transform $\mathscr{C}$ maps the line $\mathbb{R}e^{i\alpha}$ to the circle $|z - i\tan(\alpha)| = \cos(\alpha)^{-1}$, the point $z = 1$ excluded.

(c) Since the inverse Cayley transform $\mathscr{C}^{-1}(z) = i\frac{z+1}{z-1}$ is continuous in $z = -1$, we find $r > 0$ such that $\mathscr{C}^{-1}(B(-1, r)) \subset B(\mathscr{C}^{-1}(-1), \varepsilon) = B(0, \varepsilon)$. Hence we obtain

$$\begin{aligned}
\mathscr{C}(-i[\Sigma_{\frac{\pi}{2}-\delta} + \varepsilon]) &\subset \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}-\delta}\backslash B(0, \varepsilon)]) \\
&= \mathscr{C}(-i\Sigma_{\frac{\pi}{2}-\delta})\backslash \mathscr{C}(-iB(0, \varepsilon)) \\
&= \mathscr{C}(-i\Sigma_{\frac{\pi}{2}-\delta})\backslash \mathscr{C}(B(0, \varepsilon)) \\
&\subset \mathscr{C}(-i\Sigma_{\frac{\pi}{2}-\delta})\backslash B(-1, r).
\end{aligned}$$

(d) Since the Cayley transform $\mathscr{C}(z) = \frac{z-i}{z+i}$ is continuous in $z = 0$, we find some $r > 0$ such that $\mathscr{C}(B(0, r)) \subset B(\mathscr{C}(0), \varepsilon) = B(-1, \varepsilon)$. Hence we have

$$\begin{aligned}
\mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta}) \cup B(-1, \varepsilon) &\supset \mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta}) \cup \mathscr{C}(B(0, r)) \\
&= \mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta}) \cup \mathscr{C}(-iB(0, r)) \\
&= \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}+\delta} \cup B(0, r)]) \\
&\supset \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}+\delta'} - \varepsilon']) \text{ for suitable } \delta', \varepsilon' > 0. \quad \blacksquare
\end{aligned}$$

Proof of Proposition 2.1. Let $\|T^n\| \leqslant M$ for all $n \in \mathbb{N}_0$. Furthermore, let $\delta \in (0, \frac{\pi}{2})$ and $\varepsilon > 0$. By Lemma 2.3(c), we find some $r > 0$ such that

$$S := \mathscr{C}(-i\Sigma_{\frac{\pi}{2}-\delta})\backslash B(-1, r) \supset \mathscr{C}(-i[\Sigma_{\frac{\pi}{2}-\delta} + \varepsilon]).$$

Then the assertion follows from

$$\|R(\lambda, T)\| = \left\| \sum_{n=0}^{\infty} \lambda^{-n-1} T^n \right\| \leqslant M(|\lambda|-1) \qquad \text{for all} \quad |\lambda|>1$$

and the fact that $\frac{|\lambda-1|}{|\lambda|-1} \leqslant C$ for all $\lambda \in S$. The latter is clear since, by Lemma 2.3(b), there exists $\psi < \frac{\pi}{2}$ such that $\operatorname{dist}(S \backslash (1+\Sigma_\psi), \mathbb{D}) > 0$. Indeed, the estimate $\frac{|\lambda-1|}{|\lambda|-1} \leqslant C$ is then evident on $1+\Sigma_\psi$ and on $S\backslash(1+\Sigma_\psi)$. ∎

Proof of Proposition 2.2. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) is shown as Theorem 2.3 in [B1].

(c) $\Rightarrow$ (d) is trivial, (d) $\Rightarrow$ (c) is the following standard sector extension [P] for any closed and densely defined operator A in X:

$$\{\lambda R(\lambda, A); \lambda \in \Sigma_{\frac{\pi}{2}}\} \text{ is bounded}$$
$$\Rightarrow \exists \delta > 0; \{\lambda R(\lambda, A); \lambda \in \Sigma_{\frac{\pi}{2}+\delta}\} \text{ is bounded}$$

(e) $\Rightarrow$ (d) is trivial since $\mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta}) \supset \mathscr{C}(-i\Sigma_{\frac{\pi}{2}}) = \bar{\mathbb{D}}^c$.

(c) $\Rightarrow$ (e) Due to Lemma 2.3(b), the condition (c) and $\sigma(T) \subset \mathbb{D} \cup \{1\}$ combine to boundedness of $(\lambda-1)\, R(\lambda, T)$ on $\mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta'})$ for some $\delta' \in (0, \delta)$ small enough.

(f) $\Rightarrow$ (e) is trivial, (e) $\Rightarrow$ (f) follows from Lemma 2.3(d), since $-1 \in \rho(T)$ because $-1 \in \overline{\mathscr{C}(-i\Sigma_{\frac{\pi}{2}+\delta})}$ due to Lemma 2.3(b).

(d) $\Rightarrow$ (g) is trivial, (g) $\Rightarrow$ (d) follows from the maximum principle which is applicable since, if $\|T^n\| \leqslant M$ for all $n \in \mathbb{N}_0$, we have for all $|\lambda|>1$:

$$\|R(\lambda, T)\| = \left\| \sum_{n=0}^{\infty} \lambda^{-n-1} T^n \right\| \leqslant M(|\lambda|-1)^{-1}. \qquad ∎$$

2.2. *Analyticity on L_p-spaces via Interpolation*

In this section, we apply the results of the preceding section to give the proof of Theorem 1.1 saying that an operator, which is powerbounded on L_p and L_q as well as analytic on L_p, is powerbounded and analytic on L_r for all r strictly between p and q.

Proof of Theorem 1.1. Recall that $p, q \in [1, \infty]$ and $T \in \mathfrak{L}(L_p)$ is powerbounded and analytic as well as powerbounded on L_q. We fix some r strictly between p and q. Then T is powerbounded on L_r by the

Riesz–Thorin Theorem, hence, in order to establish the analyticity of T on L_r, it suffices to show by Theorem 2.2:

$$\|(\lambda-1)\,R(\lambda, T)\|_{\mathfrak{L}(L_r)} \leqslant C \qquad \text{for all} \quad |\lambda|=1, \quad \lambda\neq 1.$$

We will thus verify that $\|(\mathscr{C}(-i\mu)-1)\,R(\mathscr{C}(-i\mu), T)\|_{\mathfrak{L}(L_r)} \leqslant C$ for all $\mu\in i\mathbb{R}_+$, the argument for $\mu\in i\mathbb{R}_-$ being completely analogous. This will be achieved by Stein-interpolation between the following two properties:

$$\forall \delta_q, \varepsilon_q>0 \quad \forall \lambda\in\Sigma_{\frac{\pi}{2}-\delta_q}+\varepsilon_q\colon$$

$$\|(\mathscr{C}(-i\mu)-1)\,R(\mathscr{C}(-i\mu), T)\|_{\mathfrak{L}(L_q)} \leqslant M_0(\delta_q, \varepsilon_q) \tag{6}$$

$$\exists \delta_p, \varepsilon_p>0 \quad \forall \lambda\in\Sigma_{\frac{\pi}{2}+\delta_p}-\varepsilon_p\colon$$

$$\|(\mathscr{C}(-i\mu)-1)\,R(\mathscr{C}(-i\mu), T)\|_{\mathfrak{L}(L_p)} \leqslant M_1 \tag{7}$$

Indeed, (6) holds by Proposition 2.1 and (7) holds by Proposition 2.2. By taking a smaller value of $\delta_p>0$ we can assume

$$\delta_p<\frac{(1-\theta)\,\pi}{2\theta}, \qquad \text{where} \quad \theta\in(0,1) \text{ is given by } \frac{1}{r}=\frac{1-\theta}{q}+\frac{1}{p}.$$

Then we have $\psi_0:=\frac{\pi}{2}+\delta_p>\frac{\pi}{2}$ and $\psi_1:=\frac{\pi}{2}-\tilde{\delta}_q\in(0,\frac{\pi}{2})$, where the number $\tilde{\delta}_q$ is given by $\psi_0\theta+\psi_1(1-\theta)=\frac{\pi}{2}$. Finally, we choose $\alpha>0$ small enough such that

$$\Sigma_{\psi_0}\backslash\Sigma_{\psi_1}+\alpha\frac{z-\theta}{z-2}\subset(\Sigma_{\psi_0}-\varepsilon_p)\backslash\{1\} \qquad \text{for all} \quad z\in B,$$

where $B:=\{z\in\mathbb{C};\ \mathrm{Re}(z)\in[0,1]\}$. Now we define for all $t>0$ the function

$$\rho_t\colon B\to(\Sigma_{\psi_0}-\varepsilon_p)\backslash\{0\}, \qquad z\mapsto te^{i[\psi_1 z+\psi_0(1-z)]}+\alpha\frac{z-\theta}{z-2}.$$

We will apply Stein interpolation to the operators

$$U_z^{(t)}:=(\mathscr{C}(-i\rho_t(z))-1)\,R(\mathscr{C}(-i\rho_t(z)), T), \qquad \text{where} \quad z\in B, \quad t>0.$$

Since ρ_t maps into the sector $\Sigma_{\frac{\pi}{2}+\delta_p}-\varepsilon_p$, line (7) yields the one boundary estimate

$$\|U_{1+is}^{(t)}\|_{\mathfrak{L}(L_p)} \leqslant M_1 \qquad \text{for all} \quad r>0, \quad r\in\mathbb{R}.$$

The other boundary estimate (i.e. for the $\|U_{is}^{(t)}\|_{\mathfrak{L}(L_q)}$) will checked by using (6). For this purpose, we choose $\delta_q \in (\tilde{\delta}_q, \frac{\pi}{2})$, $\varepsilon_q > 0$ such that

$$\Sigma_{\frac{\pi}{2}-\tilde{\delta}_q} + \alpha\,\frac{is-\theta}{is-2} \subset \Sigma_{\frac{\pi}{2}-\delta_q} + \varepsilon_q \qquad \text{for all} \quad s \in \mathbb{R}.$$

The latter is possible since

$$\operatorname{Re}\left(\frac{is-\theta}{is-2}\right) \geqslant \frac{\theta}{2} \quad \text{and} \quad \left|\operatorname{Im}\left(\frac{is-\theta}{is-2}\right)\right| \leqslant \frac{2-\theta}{4} \qquad \text{for all} \quad s \in \mathbb{R}.$$

Now set $M_0 := M_0(\delta_q, \varepsilon_q)$. Since $\rho_t(is) \in \Sigma_{\frac{\pi}{2}-\delta_q} + \varepsilon_q$, we obtain from line (6):

$$\|U_{is}^{(t)}\|_{\mathfrak{L}(L_q)} \leqslant M_0 \qquad \text{for all} \quad t>0, \quad s \in \mathbb{R}.$$

Hence the Stein Interpolation Theorem yields for all $t>0$:

$$\begin{aligned} M_0^{\theta} M_1^{1-\theta} &\geqslant \|U_{\theta}^{(t)}\|_{\mathfrak{L}(L_r)} \\ &= \|(\mathscr{C}(-i\rho_t(\theta))-1)\, R(\mathscr{C}(-i\rho_t(\theta)), T)\|_{\mathfrak{L}(L_r)} \\ &= \|(\mathscr{C}(t)-1)\, R(\mathscr{C}(t), T)\|_{\mathfrak{L}(L_r)}. \end{aligned}$$

The proof is complete. ∎

3. R_q-BOUNDEDNESS

In the introduction of this paper, we recalled the notion of R-boundedness for a set $\tau \subset \mathfrak{L}(X)$, where X is a Banach space,

$$\int_0^1 \left\| \sum_j r_j(t)\, T_j f_j \right\|_X dt \leqslant C \int_0^1 \left\| \sum_j r_j(t)\, f_j \right\|_X dt \tag{8}$$

for all finite sequences (T_j) in τ and (f_j) in X. Here the r_j are the Rademacher functions. In case of $X = L_p$, one obtains by Kahane's inequality [LT2], Fubini's theorem and Khintchine's inequality [LT1]:

$$\begin{aligned} \int_0^1 \left\| \sum_j r_j(t)\, f_j \right\|_{L_p} dt &\sim \left(\int_0^1 \left\| \sum_j r_j(t)\, f_j \right\|_{L_p}^p dt \right)^{1/p} \\ &\sim \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L_p}. \end{aligned}$$

Now (8) is recognized as the following square function estimate:

$$\left\| \left(\sum_j |T_j f_j|^2 \right)^{1/2} \right\|_{L_p} \leqslant C \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L_p}.$$

This and the purpose to apply interpolation theory to R-boundedness lead Weis to the following definition [W2].

DEFINITION 3.1. Let $p,\ q \in [1, \infty]$. A set τ of sublinear bounded operators on L_p is called R_q*-bounded* if there is a constant C such that for all $N \in \mathbb{N}$, $T_1, ..., T_N \in \tau$ and $f_1, ..., f_N \in L_p$ we have

$$\left\| \left(\sum_j |T_j f_j|^q \right)^{1/q} \right\|_{L_p} \leqslant C \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L_p} \qquad \text{if} \quad 1 \leqslant q < \infty$$

$$\| \sup_j |T_j f_j| \|_{L_p} \leqslant C \, \| \sup_j |f_j| \|_{L_p} \qquad \text{if} \quad q = \infty.$$

By $R_q(\tau)$ we denote the smallest constant C for which this holds.

The following remarks are taken from [W2, (1f) and (4b)].

Remark 3.2. (a) A subset of $\mathfrak{L}(L_p)$ is bounded if and only if it is R_p-bounded.

(b) A subset of $\mathfrak{L}(L_p)$ is R-bounded if and only if it is R_2-bounded.

(c) Let $p,\ q \in (1, \infty)$. A subset τ of $\mathfrak{L}(L_p)$ is R_q-bounded if and only if $\tau' = \{T';\ T \in \tau\}$ is $R_{q'}$-bounded in $\mathfrak{L}(L_{p'})$.

The following two propositions are fundamental for the Sections 4 and 5. The first one adapts Weis' R_q-boundedness version [W2, Prop. (4b)] of the Stein interpolation theorem to our purposes.

PROPOSITION 3.3. *Let $p \in [1, \infty]$ and (e^{tA}) a bounded analytic semigroup on L_p. Furthermore, let $q,\ q_0 \in [1, \infty]$ be such that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{p}$ for some $\theta \in (0, 1]$.*

(a) *For all $\delta < \frac{\pi}{2}$, let $\{\lambda R(\lambda, A);\ \lambda \in \Sigma_\delta\}$ be R_{q_0}-bounded. Then $\{\lambda R(\lambda, A);\ \lambda \in i\mathbb{R},\ \lambda \neq 0\}$ is R_q-bounded.*

(b) *Let $\{e^{tA};\ t > 0\}$ be R_{q_0}-bounded. Then $\{e^{zA};\ |\arg(z)| = \delta_q\}$ is R_q-bounded for some $\delta_q > 0$.*

Proof. By analyticity and Remark 3.2(a), there exists $\delta_1 \in (0, \frac{(1-\theta)\pi}{2\theta})$ such that

$$\{e^{zA};\ |\arg(z)| \leqslant \delta_1\} \qquad \text{and}$$

$$\left\{\lambda R(\lambda, A);\ |\arg(\lambda)| \leqslant \frac{\pi}{2} + \delta_1\right\} \text{ are } R_p\text{-bounded.}$$

Hence we can assume $\theta \in (0, 1)$. Then $\{e^{zA};\ |\arg(z)| = \theta\delta_1\}$ is R_q-bounded due to [W2, Prop. (4b)], and (b) is proved. But assertion (a) follows also from [W2, Prop. (4b)] since

$$\{\lambda R(\lambda, A);\ |\arg(\lambda)| \leqslant \delta_0\} \text{ is } R_{q_0}\text{-bounded,}$$

where $\delta_0 \in (0, \frac{\pi}{2})$ is given by $\frac{\pi}{2} = (1-\theta)\,\delta_0 + \theta(\frac{\pi}{2} + \delta)$. ∎

In all our applications, we will verify the hypotheses of the preceding interpolation result by the aid of the following result of Fefferman and Stein [FS] on R_q-boundedness on spaces (Ω, μ, d) of homogenous type as defined e.g. in [CD]. We will use the standard notations $B(x, r)$ for the ball centered at x with radius r and $V(x, r) := \mu(B(x, r))$.

We consider the Hardy–Littlewood maximal operator M defined by

$$Mf(x) := \sup_{r>0} V(x, r)^{-1} \int_{B(x,r)} |f(y)|\, d\mu(y),$$

which is well-known to be bounded on $L_p(\Omega)$ for all $p \in (1, \infty]$.

Proposition 3.4. *Let (Ω, μ, d) be a space of homogeneous type. Then the Hardy–Littlewood maximal operator M on Ω satisfies*:

$$\{M\} \textit{ is } R_q\textit{-bounded on } L_p(\Omega) \textit{ for all } q \in (1, \infty],\ p \in (1, \infty).$$

Proof. This is shown in [FS] for the case $\Omega = \mathbb{R}^n$, and the proof given there extends easily to spaces of homogeneous type. ∎

4. R_q-BOUNDEDNESS VIA DOMINATION BY THE HARDY–LITTLEWOOD MAXIMAL OPERATOR

In this section, (Ω, μ, d) is again a space of homogenous type. By the Fefferman–Stein result Proposition 3.4, the Hardy–Littlewood maximal operator M on Ω has the following property: $\{M\}$ is R_q-bounded on $L_p(\Omega)$ for all $q \in (1, \infty]$, $p \in (1, \infty)$. Hence, the observation that, by definition, R_q-boundedness is preserved by domination, establishes already the following main tool of this section.

Remark 4.1. Let $p_0 \in [1, \infty]$ and $\tau \subset \mathfrak{L}(L_{p_0}(\Omega))$ satisfy

$$|Sf| \leqslant CMf \qquad \text{a.e.} \quad \text{for all } \; S \in \tau, \, f \in L_{p_0}(\Omega). \tag{9}$$

Then τ is R_q-bounded on $L_p(\Omega)$ for all $q \in (1, \infty]$, $p \in (1, \infty)$.

The following important lemma is shown in [DR, Prop. 2.4].

LEMMA 4.2. *Let Ω be a space of homogeneous type and dimension $D > 0$, i.e.*

$$V(x, \lambda r) \leqslant C\lambda^D V(x, r) \qquad \textit{for all} \quad x \in \Omega, \, \lambda \geqslant 1, \, r > 0.$$

Let $g: \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function satisfying $\sum_k k^{D-1} g(k^m) < \infty$ for some $m > 0$. Now define the Poisson kernels $G_t(x, y) := V(x, t^{1/m})^{-1} g(d(x, y)^m \, t^{-1})$. Then the corresponding integral operators $G_t f(x) = \int G_t(x, y) \, f(y) \, d\mu(y)$ are uniformly dominated by the Hardy–Littlewood maximal operator M, i.e.

$$|G_t f| \leqslant C' Mf \qquad \textit{a.e.} \quad \textit{for all} \; t > 0, \, p \in [1, \infty], \, f \in L_p(\Omega).$$

Remark 4.1 and Lemma 4.2 together provide a powerful method to verify R_q-boundedness on spaces Ω of homogeneous type. It was inspired by ideas in [W2], for the special case $\Omega \subset \mathbb{R}^n$, $q = 2$ a similar method is used in [CP, Section 4].

4.1. *Maximal Regularity via Poisson Bounds*

Now we apply the preceding observations in order to verify (discrete) maximal regularity of integral operators satisfying Poisson bounds. We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. By considering the zero-extension of the kernels $D^k p_n$ to Ω_1^2 otherwise, we can assume $\Omega = \Omega_1$. As in Lemma 4.2, we denote again the Poisson kernels $G_t(x, y) := V(x, t^{1/m})^{-1} \, g(d(x, y)^m \, t^{-1})$, $t > 0$. For the corresponding integral operators $G_t f(x) = \int G_t(x, y) \, f(y) \, d\mu(y)$ we obtain from the Poisson bound (3) and Lemma 4.2:

$$|(T - I)^k \, T^n f| \leqslant n^{-k} G_n \, |f| \leqslant C n^{-k} Mf \qquad \text{for all} \quad k = 0, 1, \quad n \in \mathbb{N}$$

This implies directly the powerboundedness and the analyticity of T on L_p for all $p \in (1, \infty)$. Furthermore, we conclude from the Fefferman–Stein result in form of Remark 4.1 that

$$\{n^k (T - I)^k \, T^n; \, k = 0, 1, \, n \in \mathbb{N}\} \text{ is } R\text{-bounded on } L_p \text{ for all } p \in (1, \infty). \tag{10}$$

Thus we obtain the discrete maximal regularity of T on L_p for all $p \in (1, \infty)$ from [B1, Theorem 1.1], cited as Theorem A in the introduction of this paper. ∎

Similar arguments yield a short proof and a slight improvement of the result for the continuous time setting due to Coulhon and Duong [CD, Theorem 1.2], see also [HP].

THEOREM 4.3. *Let Ω be a measurable subset of a space (Ω_1, d, μ) of homogenous type and dimension $D > 0$. Let (e^{tA}) be a bounded analytic semigroup on $L_2(\Omega)$ which has an integral kernel $P_t(x, y)$ satisfying the following Poisson bounds*:

$$|P_t(x, y)| \leqslant (V_{\Omega_1}(x, t^{1/m})^{-1} \wedge V_{\Omega_1}(y, t^{1/m})^{-1})\, g(d(x, y)^m\, t^{-1}) \tag{11}$$

for all $t > 0$, some $m > 0$ and a bounded decreasing function g: $\mathbb{R}_+ \to \mathbb{R}_+$ with

$$r^{D+\delta} g(r^m) \leqslant C \qquad \textit{for all} \quad r \in \mathbb{R}_+ \textit{ and some } \delta > 0.$$

Then, for all $p \in (1, \infty)$, (e^{tA}) is a bounded analytic semigroup on $L_p(\Omega)$ and A has maximal regularity on $L_p(\Omega)$.

In [CD] the same result was obtained under the following stronger growth condition on the function g:

$$r^{2D+\delta} g(r^m) \leqslant C \qquad \text{for all} \quad r \in \mathbb{R}_+ \text{ and some } \delta > 0.$$

Proof. According to [DR, Prop. 2.3], (e^{tA}) acts as a bounded strongly continuous semigroup on $L_p(\Omega)$ for all $p \in [1, \infty)$. Hence, due to the analyticity on $L_2(\Omega)$, the analyticity on $L_p(\Omega)$ for $p \in (1, \infty)$ follows by interpolation.

For the proof of the second assertion, we fix some $p \in (1, \infty)$. By considering the zero-extension of the kernels P_t to Ω_1^2 otherwise, we can assume $\Omega = \Omega_1$.

We denote again the Poisson kernels $G_t(x, y) := V(x, t^{1/m})^{-1}\, g(d(x, y)^m\, t^{-1})$, $t > 0$. For the corresponding integral operators $G_t f(x) = \int G_t(x, y) f(y)\, d\mu(y)$ we obtain from the Poisson bound (11) and Lemma 4.2:

$$|e^{tA} f| \leqslant G_t\, |f| \leqslant CMf \qquad \text{for all} \quad t > 0.$$

The Fefferman–Stein result in form of Remark 4.1 yields

$$\{e^{tA};\, t > 0\} \text{ is } R_q\text{-bounded for all } q \in (1, \infty).$$

We apply this for some $q \in (1, \infty) \setminus \{2\}$ such that 2 is between p and q, and obtain by our interpolation Proposition 3.3(b) as well as Remark 3.2(b):

$$\{e^{zA};\ |\arg(z)| = \delta\} \text{ is } R\text{-bounded for some } \delta > 0.$$

Hence A has maximal regularity due to Weis' characterization [W1, Theorem 4.2]. ∎

5. R_∞-BOUNDEDNESS VIA DOMINATION AND TRANSFERENCE

Let $\mathcal{G}$ be a LCA group, $p \in [1, \infty)$ and $R\colon \mathcal{G} \to \mathfrak{L}(L_p)$ a strongly continuous bounded representation of $\mathcal{G}$, say with bound M. We consider the usual class of transference kernels

$$\mathcal{K} := \{k \in L_1(\mathcal{G});\ \operatorname{supp}(k) \text{ compact}\}$$

and the corresponding transferred operators

$$H_k := \int_{\mathcal{G}} k(u)\, R_{-u}\, du, \qquad k \in \mathcal{K}.$$

Then the standard transference theorem of Coifman and Weiss [CW, Theorem 2.4] says

$$\|H_k\|_{\mathfrak{L}(L_p)} \leqslant M^2\, \|T_k\|_{\mathfrak{L}(L_p(\mathcal{G}))} \qquad \text{for all} \quad k \in \mathcal{K},$$

where T_k denotes the operator of convolution with k. Here we show for the case of a positive representation the more general result that not only the boundedness of a single operator can be transferred, but even the R_∞-boundedness of a set of operators. Our following result is based on [CW, Section 4].

THEOREM 5.1. *Let $\mathcal{K}_0 \subset \mathcal{K}$ be such that $\{T_k; k \in \mathcal{K}_0\}$ is R_∞-bounded on $L_p(\mathcal{G})$. If R_u is a positive operator for all $u \in \mathcal{G}$ then $\{H_k; k \in \mathcal{K}_0\}$ is R_∞-bounded on L_p. More precisely,*

$$R_\infty(\{H_k; k \in \mathcal{K}_0\}) \leqslant M^2 R_\infty(\{T_k; k \in \mathcal{K}_0\}).$$

Proof. Our first step is to show

$$\|\max_j |H_{k_j} f_j|\,\| \leqslant M\, \|\max_j |H_{k_j}(R_v f_j)|\,\|_p \qquad \text{for all} \quad v \in \mathcal{G} \tag{12}$$

and all finite sequences $k_1, ..., k_N \in \mathscr{K}_0, f_1, ..., f_N \in L_p$. But this is clear since for all fixed $j_0 \in \{1, ..., N\}$ we have

$$|H_{k_{j0}}(R_v f_{j0})| = |R_v(H_{k_{j0}} f_{j0})| \leqslant R_v(|H_{k_{j0}} f_{j0}|) \leqslant R_v(\max_j |H_{k_j} f_j|),$$

where we used the positivity of the operator R_v twice. Now let $V \subset \mathscr{G}$ be any set of finite measure and K the union over all $\operatorname{supp}(k_j)$. By making use of (12) for the first step, we obtain for all $v \in V$:

$$\begin{aligned}\|\max_j |H_{k_j} f_j| \|_p^p &\leqslant M^p \|\max_j |H_{k_j}(R_v f_j)| \|_p^p \\ &= M^p \left\| \max_j \left| \int_K k_j(u)\, R_{v-u} f_j \, du \right| \right\|_p^p \\ &= M^p \left\| \max_j \left| \int_{\mathscr{G}} k_j(u)\, \chi_{V-K}(v-u)\, R_{v-u} f_j \, du \right| \right\|_p^p \\ &= M^p \|\max_j |k_j * (\chi_{V-K} R_{(\cdot)} f_j)(v)| \|_p^p.\end{aligned}$$

Thus, integration over V yields the first step in the following estimate, where we denote $\chi := \chi_{V-K}$ and $\tau := \{T_k; k \in \mathscr{K}_0\}$:

$$\begin{aligned}\|\max_j |H_{k_j} f_j| \|_p^p &\leqslant M^p\, |V|^{-1} \int_V \|\max_j |k_j * (\chi R_{(\cdot)} f_j)(v)| \|_p^p \, dv \\ &\leqslant M^p\, |V|^{-1} \int \|\max_j |k_j * (\chi [R_{(\cdot)} f_j](\omega))| \|_{L_p(\mathscr{G})}^p \, d\omega \\ &\leqslant M^p\, |V|^{-1}\, R_\infty(\tau)^p \int \|\max_j |\chi [R_{(\cdot)} f_j](\omega)| \|_{L_p(\mathscr{G})}^p \, d\omega \\ &= M^p\, |V|^{-1}\, R_\infty(\tau)^p \int_{V-K} \|\max_j |R_v f_j| \|_p^p \, dv \\ &\leqslant M^p\, |V|^{-1}\, R_\infty(\tau)^p \int_{V-K} \|R_v(\max_j |f_j|)\|_p^p \, dv \\ &= M^{2p} R_\infty(\tau)^p\, |V-K|\, |V|^{-1} \|\max_j |f_j| \|_p^p.\end{aligned}$$

Since every LCA $\mathscr{G}$ group is amenable in the sense of [CW], the claim follows by optimization over V. ∎

The importance of the following example for the case $\mathscr{G} = \mathbb{Z}$ is evident in view of the formula $\sum_{n=0}^{\infty} \frac{\lambda-1}{\lambda^{n+1}} T^n = (\lambda-1)\, R(\lambda, T)$.

EXAMPLE 5.2. For all $\lambda > 1$, let

$$k_\lambda(n) := \begin{cases} \dfrac{\lambda - 1}{\lambda^{n+1}} & n \geqslant 0 \\ 0 & n < 0. \end{cases}$$

Then $\{T_{k_\lambda}; \lambda > 1\}$ is R_q-bounded on l_p for all $q \in (1, \infty]$ and $p \in (1, \infty)$.

Proof. This follows from the Fefferman–Stein result in form of Remark 4.1 and our domination Lemma 4.2 since the k_λ satisfy the following Poisson bounds:

$$|k_{\exp(\mu^{-1})}(n-m)| \leqslant \mu^{-1} \exp\left(-\frac{|n-m|}{\mu} \right) \qquad \text{for all} \quad \mu > 0. \quad \blacksquare$$

COROLLARY 5.3. *Let $p \in (1, \infty)$ and $T \in \mathfrak{L}(L_p)$ be a subpositive contraction. Then, for all $\delta < \frac{\pi}{2}$, the set $\{(\lambda - 1)\, R(\lambda, T); \lambda \in 1 + \Sigma_\delta\}$ is R_∞-bounded on L_p.*

Proof. By subpositivity, there exists a positive contraction $S \in \mathfrak{L}(L_p)$ such that $|Tf| \leqslant S\,|f|$ for all $f \in L_p$. From the observation

$$\begin{aligned} |(\lambda-1)\, R(\lambda, T)\, f| &= \left| (\lambda - 1) \sum_{m=0}^{\infty} \lambda^{-m-1} T^m f \right| \\ &\leqslant |\lambda - 1| \sum_{m=0}^{\infty} |\lambda|^{-m-1}\, S^m\, |f| \\ &= |\lambda - 1|\, R(|\lambda|, S)\, |f| \\ &\leqslant C_\delta (|\lambda| - 1)\, R(|\lambda|, S)\, |f| \qquad \text{for all} \quad \lambda \in 1 + \Sigma_\delta \end{aligned}$$

and the fact that R_q-boundedness is preserved by domination, we conclude that it suffices to show:

$$\{(\lambda - 1)\, R(\lambda, S); \lambda > 1\} \text{ is } R_\infty\text{-bounded on } L_p. \tag{13}$$

Since S is a positive contraction, due to the dilation theorem of Akcoglu and Sucheston [AS] we can assume that S is invertible and that also S^{-1} is a positive contraction. Hence (13) follows from Theorem 5.1, applied to the representation $R\colon \mathbb{Z} \to \mathfrak{L}(L_p)$, $u \mapsto S^{-u}$ and the kernels k_λ of Example 5.2, since we have

$$H_{k_\lambda} = \sum_{u \in \mathbb{Z}} k_\lambda(u)\, R_{-u} = (\lambda - 1)\, R(\lambda, S). \quad \blacksquare$$

For later use we consider a second example.

EXAMPLE 5.4. For all $\lambda > 0$, let

$$k_\lambda(t) := \begin{cases} \lambda e^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0. \end{cases}$$

Then $\{T_{k_\lambda}; \lambda > 0\}$ is R_q-bounded on $L_p(\mathbb{R})$ for all $q \in (1, \infty]$ and $p \in (1, \infty)$.

Proof. This follows from the Fefferman–Stein result in form of Remark 4.1 and our domination Lemma 4.2 since the k_λ satisfy the following Poisson bounds:

$$|k_{\mu^{-1}}(x-y)| \leqslant \mu^{-1} \exp\left(-\frac{|x-y|}{\mu}\right) \qquad \text{for all} \quad \mu > 0. \quad \blacksquare$$

5.1. *Maximal Regularity via Transference*

In this section, we combine our transference principle for R_∞-boundedness in form of Corollary 5.3 and the interpolation result Proposition 3.3 in order to verify maximal regularity. We begin with the proof of Theorem 1.3.

Proof of Theorem 1.3. By Remark 3.2(b) and [B1, Theorem 1.1], cited as Theorem A in the introduction of this paper, the discrete maximal regularity of the subpositive analytic contraction $T \in \mathfrak{L}(L_p)$ is equivalent to the fact that

$$\{(\lambda - 1)\, R(\lambda, T); \lambda \in 1 + i\mathbb{R}, \lambda \neq 1\} \text{ is } R_2\text{-bounded on } L_p. \tag{14}$$

Since, by Corollary 5.3, the set $\{(\lambda-1)\, R(\lambda, T); \lambda \in \Sigma_\delta\}$ is R_∞-bounded on L_p for all $\delta < \frac{\pi}{2}$, line (14) follows by applying our interpolation Proposition 3.3(a) to $A := T - I$ and $\theta := \frac{p}{2}$, provided that $p \in (1, 2]$.

For $p \in [2, \infty)$, line (14) is obtained by duality and Remark 3.2(c) since T' is a subpositive analytic contraction on $L_{p'}$. $\blacksquare$

Remark. In order to illustrate our method also in the continuous time situation, we give an alternative proof of the Weis/Lamberton result that, if (e^{tA}) is a subpositive analytic contraction semigroup on L_p, $p \in (1, \infty)$, then A has maximal regularity.

By duality, it suffices to consider the case $p \in (1, 2]$. We want to verify the equivalent resolvent condition "$\{\lambda R(\lambda, A);\ \lambda \in i\mathbb{R},\ \lambda \neq 0\}$ is R_2-bounded" [W1, Cor. 4.4]. Using interpolation in form of Proposition 3.3(a) [applied to $\theta := \frac{p}{2}$], it is sufficient to show that $\{\lambda R(\lambda, A); \lambda \in \Sigma_\delta\}$ is R_∞-bounded for all $\delta < \frac{\pi}{2}$.

Due to Fendler's semigroup version of the dilation theorem [F], we can now assume that (e^{tA}) is a group of positive contractions on L_p. Hence, in view of the estimate

$$\begin{aligned}|\lambda R(\lambda, A)\, f| &\leqslant |\lambda| \int_0^\infty e^{-Re(\lambda)\, t} e^{tA}\, dt\, |f| \\ &= |\lambda|\; R(\mathrm{Re}(\lambda), A)\, |f| \\ &\leqslant C_\delta\, \mathrm{Re}(\lambda)\; R(\mathrm{Re}(\lambda), A)\, |f| \qquad \text{for all} \quad \lambda \in \Sigma_\delta,\end{aligned}$$

we finally have to show that $\{\lambda R(\lambda, A);\ \lambda > 0\}$ is R_∞-bounded. But this follows directly from Theorem 5.1, applied to the kernels k_λ of Example 5.4, since

$$\int_{\mathbb{R}} k_\lambda(t)\, e^{tA}\, dt = \lambda R(\lambda, A).$$

5.2. *A Proof of the Maximal Ergodic Theorem via Transference*

As a further illustration of our transference principle for R_∞-boundedness, we give a proof of the well-known fact that the Maximal Ergodic Theorem can be viewed as the transferred Hardy–Littlewood maximal theorem. For the discrete case this is shown e.g. in [CW, Theorem 4.4].

THEOREM 5.5 (Maximal Ergodic Theorem). *Let $p \in (1, \infty)$.*

(a) *Let $(e^{tA})_{t \in \mathbb{R}}$ be a positive bounded C_0-group on L_p. Then*

$$\left\| \sup_{t>0} \left| t^{-1} \int_0^t e^{sA} f\, ds \right| \right\|_{L_p} \leqslant C\, \|f\|_{L_p}.$$

(b) *Let $T \in \mathfrak{L}(L_p)$ be invertible such that T, T^{-1} are positive and powerbounded. Then*

$$\left\| \sup_{j \geqslant 0} \left| \frac{1}{j+1} \sum_{n=0}^{j} T^n f \right| \right\|_{L_p} \leqslant C\, \|f\|_{L_p}.$$

Proof. (a) Consider the kernels $k_t := t^{-1} \chi_{[0,\, t]}$ and the representation $R\colon \mathbb{R} \to \mathfrak{L}(L_p)$, $s \mapsto e^{-sA}$. Since the corresponding transferred operators H_{k_t} are given by

$$H_{k_t} = \int_r k_t(s)\, R_{-s}\, ds = t^{-1} \int_0^t e^{sA}\, ds,$$

it suffices to show that $\{H_{k_t};\ t>0\}$ is R_∞-bounded on L_p. Hence, by our transference principle Theorem 5.1, it is enough to show that the corresponding convolution operators $\{T_{k_t};\ t>0\}$ are R_∞-bounded on $L_p(\mathbb{R})$. But this is the boundedness of the left-sided Hardy–Littlewood maximal operator M_- for $\mathbb{R}$ defined by

$$M_-f(x) := \sup_{t>0} t^{-1} \int_0^t |f(x-s)|\, ds = \sup_{t>0} T_{k_t}\, |f|\, (x).$$

The proof of (a) is complete, and the proof of (b) is analogous. ∎

Remark. Supposing in (a) additionally the analyticity of (e^{tA}), Weis shows in [W2] the maximal regularity of A as an application of the Maximal Ergodic Theorem. Indeed, by duality, we can assume $p \in (1, 2]$. Due to analyticity, there exists $\delta>0$ such that the function $M\colon \Sigma_\delta \to \mathfrak{L}(L_p)$, $z \mapsto z^{-1} \int_0^z e^{yA}\, dy$ is bounded and analytic. Hence, according to Remark 3.2(a) and the Maximal Ergodic Theorem 5.5(a), the following two conditions are satisfied:

$$\{M(z);\ z \in \Sigma_\delta\} \text{ is } R_p\text{-bounded on } L_p.$$

$$\{M(t);\ t>0\} \text{ is } R_\infty\text{-bounded on } L_p.$$

By interpolation [W2, Prop. (4b)] and then applying [W1, Prop. 2.8(b)], we obtain:

$$\{M(z);\ z \in \Sigma_{\frac{p}{2}\delta}\} \qquad \text{and} \qquad \{zM'(z);\ z \in \Sigma_{\frac{1}{2}\delta}\} \text{ are } R\text{-bounded on } L_p.$$

But since $e^{zA} = M(z) + zM'(z)$, we deduce that $\{e^{zA};\ z \in \Sigma_{1/2\delta}\}$ is R-bounded, thus A has maximal regularity due to Weis' characterization [W1, Theorem 4.2].

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